Phase modulation is used to estimate parameters for sinusoidal frequency-modulated signals.

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***Abstract*— We present in this correspondence a new estimator for the sinusoidal frequency-modulated (SFM) signals' phase parameters. In order to create a new modulated signal, we first created a straightforward function to conduct phase modulation on the original SFM signal. Second, we determined the modulated signal's amplitude spectrum (AS) and demonstrated that it had periodic peaks. Third, by identifying the AS peaks, we were able to estimate the parameters of the SFM signals. In order to lessen bias, the estimations were finally improved. The suggested method has a lower signal-to-noise ratio threshold than the current approaches, according to the simulation findings.**

***Index Terms*—Phase modulation, parameter estimation, sinusoidal frequency-modulated signals.**

I. Introduction

SINUSOIDAL frequency-modulated (SFM) signals arise in many signal processing applications [1]–[5]. The most notable application is in micro-Doppler (m-D) signals [6], [7]. A common model of m-D signals is an SFM one, coming from the fact that rotating and/or vibrating parts of targets cause m-D signals in the form close to such a model [1]. Therefore, the parameter estimation of SFM signals is very useful for identifying targets of interest (i.e., helicopters, ships, or aircraft with rotating antennas) [6], [8]. In recent years, many algorithms for estimating SFM parameters have been published.

The traditional maximum-likelihood estimation can be used to estimate the parameters of SFM signals. However, it yields a multidimensional search and is computationally prohibited from practical applications [9]. Wang *et al.* [10] proposed an instantaneous phase-based method using a phase unwrapping technique followed by a nonlinear coupled least square (PULS) method. It was shown that the PULS method is unbiased, and its estimation performance can approach the Cramér-Rao lower bound (CRLB) at a high signal-to-noise ratio (SNR). However, the PULS method exhibits a high SNR threshold due to the phase unwrapping step [11].

The instantaneous frequency (IF) curves of SFM signals are still sinusoidal functions. Therefore, it seems intuitively appealing to obtain estimations for SFM signals by forming an IF estimate from the peak of a time-frequency distribution (TFD) [12], [13]. Aside from the IF, instantaneous frequency rate (IFR) or chirp-rate (CR) [14]–[16] is also be used to obtain the parameter estimation of SFM signals. In [9], the method develops a local high-order phase function and estimates parameters of SFM signals from peak locations in the time-frequency rate domain. However, methods for calculating TFD or time-CR representation [14] have an intensive computation load. Therefore, these estimators based on TFD or time-CR representation also suffer a heavy computational burden and are limited to handle short observed signals.

Recently, Igor Djurovic´ *et al.* proposed the quasi-maximum-likelihood (QML) estimator [17]–[19] and then extended it in [1] to estimate the SFM parameters using a two-step procedure. In the first step, the signal parameters are estimated from the IF obtained by the short-time Fourier transform (STFT)-based estimator. In the second stage, these estimates are refined by the residual phase. The refinement procedure is performed several times to produce the mean square error (MSE) on the CRLB and remove any residual bias from the estimates. This process is repeated with different window sizes for the STFT, and the estimate that maximizes the QML parameter is chosen [20]. The method has better performance than the existing methods in terms of the SNR threshold. However, this method resorts to repeatedly performing the STFT for various window widths, which results in a very intensive computation burden.

In this letter, we focus on the estimation of the phase parameters of SFM signals at low SNRs. First, we develop a simple function to perform phase modulation on the original SFM signal to obtain a new modulated signal. Second, we derive the amplitude spectrum (AS) of the modulated signal and prove that the AS exhibits peaks periodically. Third, we obtain the parameter estimation of SFM signals by locating the AS peaks. Finally, the estimates are refined by the refinement strategy in [1] to achieve the CRLB. Since the AS peaks are robust to the influence of noise, the proposed method outperforms the existing methods in terms of the SNR threshold, confirmed by Monte Carlo simulations.

The remainder of the letter is organized as follows: The proposed estimator for SFM signals is introduced in Section II. The estimating phase modulation coefficient procedure is given in Section III. The computational complexity is discussed in Section IV. The simulation results and conclusions are presented in Sections V and VI, respectively.

II. Proposed Estimator

A. Signal Model

The following model can describe SFM signals [1]:

$$x(t) = \exp[j\vartheta(t)] = \exp(ja\sin(bt+c)),\ 0 \le t \le T \quad (1)$$

where T is the signal duration, $\vartheta(t) = a\sin(bt + c)$ is the phase function, $a > 0$is the so-called modulation index, b is the radian frequency, and c is the initial phase and limited in the interval $[-\pi, \pi)$ for avoiding modulo 2π. In this letter, we focus on the estimation of the phase parameters $\mathbf{a} = [a\ b\ c]^T$ of SFM signals.

B. Phase Modulation

We first present a phase modulation function (PMF), which is defined as follows:

$$g(t) = \exp\left(j\frac{Dt^2}{2}\right),\ 0 \le t \le T \quad (2)$$

We assume that D is a known constant and referred to as the phase modulation coefficient (PMC). However, D is unknown in practice and is required to be estimated. We will discuss this problem further in Section III. It is clear that the PMF is a linear frequency modulation (LFM) signal. For SFM signals, we can employ the PMF to perform the phase modulation on the original SFM signal according to the following formula

$$\begin{aligned} z(t) &= x(t)\,g(t) \\ &= \exp(ja\sin(bt+c))\exp\left(j\frac{Dt^2}{2}\right) \\ &= \exp\left(j\left(a\sin(bt+c) + \frac{Dt^2}{2}\right)\right) \\ &= \exp(j\Phi(t)) \end{aligned} \quad (3)$$

where $\Phi(t) = a\sin(bt+c) + Dt^2/2$. The modulated signal $z(t)$ is a hybrid linear frequency modulation-sinusoidal frequency modulation signal [11]. Note that the added phase term $Dt^2/2$ is introduced to estimate the phase parameters of SFM signals.

The Fourier transform of $z(t)$ is as follows:

$$\begin{aligned} Z(j\omega) &= \int_{-\infty}^{+\infty} e^{j\Phi(t)} e^{-j\omega t} dt \\ &= \int_{-\infty}^{+\infty} e^{j[\Phi(t)-\omega t]} dt = \int_{-\infty}^{+\infty} e^{j\varphi(t,\omega)} dt \end{aligned} \quad (4)$$

where $\varphi(t,\omega) = \Phi(t) - \omega t$. The Fourier transform of $z(t)$ is relatively complicated. A very useful and much simpler approximation can be derived using the principle of stationary phase (PSP) [21]. According to the PSP, the stationary points are solutions of the following [21]:

$$\frac{\partial\varphi(t,\omega)}{\partial t} = \Phi'(t) - \omega = ab\cos(bt+c) + Dt - \omega = 0 \quad (5)$$

where $\Phi'(t)$ is the first time derivative of $\Phi(t)$. In general, (5) has multiple roots. However, when $D > ab^2$, we can obtain

$$\Phi''(t) = -ab^2\sin(bt+c) + D > 0 \quad (6)$$

where $\Phi''(t)$ is the second time derivative of $\Phi(t)$. In this case, (5) has only one root because $\Phi'(t)$ is a strictly monotonically increasing function due to $\Phi''(t) > 0$. Therefore, (5) has only one stationary point t_ω

$$ab\cos(bt_\omega + c) + Dt_\omega - \omega = 0 \quad (7)$$

Note that t_ω denotes the localization of ω in the time domain, and it is a function of ω. Thus, the PSP approximation to the spectrum of $z(t)$ is [21]

$$Z(j\omega) \approx \sqrt{\frac{2\pi}{\Phi''(t_\omega)}} \exp\left(j\left(\Phi(t_\omega) - \omega t_\omega + \frac{\pi}{4}\right)\right),\ \omega_0 \le \omega \le \omega_0 + B \quad (8)$$

where ω_0 denotes the initial frequency and B denotes the bandwidth of the modulated signal. By (6), we can obtain $\Phi''(t) > 0$ when $D > ab^2$. Therefore, substituting (6) into (8), the AS of $z(t)$ is

$$\begin{aligned} |Z(j\omega)| &\approx \sqrt{\frac{2\pi}{\Phi''(t_\omega)}} \\ &= \sqrt{\frac{2\pi}{-ab^2\sin(bt_\omega + c) + D}} \end{aligned} \quad (9)$$

C. Parameter Estimation

When $D > ab^2$, according to (9), we can conclude that, in general, there exist many points $\{t_{\omega_i} | i = 1, 2, \ldots\}$ such that

$$\sin(bt_{\omega_i} + c) = 1 \quad (10)$$

These points maximize $|Z(j\omega)|$ at the frequency $\{\omega_i | i = 1, 2, \ldots\}$, and the maximum of $|Z(j\omega)|$ is

$$R = \max(|Z(j\omega)|) \approx \sqrt{\frac{2\pi}{-ab^2 + D}} \quad (11)$$

Since the sine function is periodic, $|Z(j\omega)|$ shows peaks periodically in the frequency interval $[\omega_0, \omega_0 + B]$. For example, we consider an SFM signal. After phase modulation, the AS of the modulated signal is shown in Fig. 1. We can see that the AS exhibits peaks at the frequencies ω_1, ω_2, and ω_3, and the frequency difference of any two adjacent peaks is equal.

Having periodic peaks is a crucial property of the modulated signals. Using this property, we can easily obtain the parameter estimation of SFM signals by locating the AS peaks. This is why we take phase modulation to the SFM signals. Now, we start proving this property. Without loss of generality, assume that we have located any two adjacent peaks of the AS at the frequencies ω_i and ω_{i+1}. Since ω_i and ω_{i+1} satisfy (7), we can obtain

$$\begin{aligned} ab\cos(bt_{\omega_i} + c) + Dt_{\omega_i} &= \omega_i \\ ab\cos(bt_{\omega_{i+1}} + c) + Dt_{\omega_{i+1}} &= \omega_{i+1} \end{aligned} \quad (12)$$

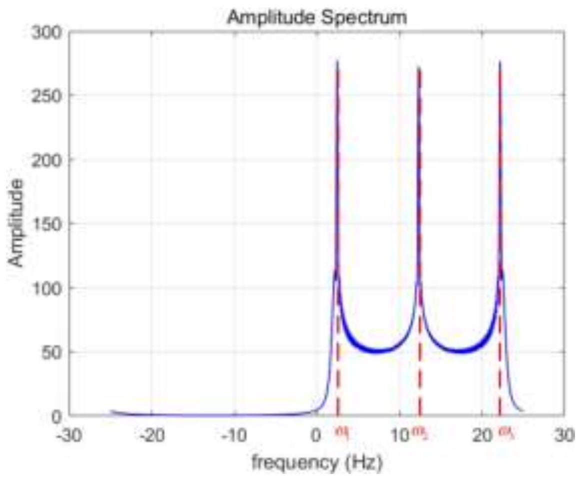

Fig. 1. AS of the modulated signal. (The parameters are $a = [10\pi \ \ 0.1\pi \ \ 0]$, $D = 3.101$, $t \in [0, \ 50]$ s, and $T_s = 0.02$ s).

In addition, since (10) holds at the frequencies ω_i and ω_{i+1}, it is easy to deduce

$$\begin{cases} \cos(bt_{\omega_i} + c) = 0 \\ \cos(bt_{\omega_{i+1}} + c) = 0 \end{cases} \tag{13}$$

Substituting (13) into (12) yields

$$\begin{cases} Dt_{\omega_i} = \omega_i \\ Dt_{\omega_{i+1}} = \omega_{i+1} \end{cases} \tag{14}$$

Then, the frequency difference of any two adjacent peaks can be expressed as

$$\Delta\omega_{i+1} = \omega_{i+1} - \omega_i \tag{15}$$

Substituting (14) into (15) gives

$$\Delta\omega_{i+1} = Dt_{\omega_{i+1}} - Dt_{\omega_i} = D(t_{\omega_{i+1}} - t_{\omega_i}) \tag{16}$$

Since (10) also holds at the times $t_{\omega_{i+1}}$ and t_{ω_i}, we have

$$t_{\omega_{i+1}} - t_{\omega_i} = 2\pi/b \tag{17}$$

Substituting (17) into (16) gives

$$\Delta\omega_{i+1} = 2\pi D/b \tag{18}$$

(18) proves that the frequency difference of any two adjacent peaks is equal. Therefore, the AS peaks of the modulated signal are periodic, with a period of $2\pi D/b$. By (18), we can obtain

$$b = 2\pi D/\Delta\omega_{i+1} = 2\pi D/(\omega_{i+1} - \omega_i) \tag{19}$$

(19) demonstrates that if the PMC D is known, then we can estimate the parameter b from the AS peak locations of the modulated signal.

Next, by (11), we can obtain

$$a \approx D\, b^2 - 2\pi \ R^2 b^2 \tag{20}$$

(20) states that the parameter a can be estimated from the amplitude of peaks.

Finally, by (10), we can obtain

$$bt_{\omega_i} + c = \arcsin(1) = \pi/2 + 2\pi(i - 1) \tag{21}$$

By (14), we have

$$t_{\omega_i} = \omega_i/D \tag{22}$$

Substituting (22) into (21) yields

$$c = \pi/2 + 2\pi(i - 1) - \omega_i b/D \tag{23}$$

In summary, by locating the AS peaks of the modulated signals, we can obtain parameter estimation of the original SFM signal by the following formula

$$\begin{cases} b = 2\pi D/(\omega_{i+1} - \omega_i) \\ a \approx D\, b^2 - 2\pi \ R^2 b^2 \\ c = \pi/2 + 2\pi(i - 1) - \omega_i b/D \end{cases} \tag{24}$$

Remake: Recalling (19), to estimate the parameter b, we need to obtain at least two AS peaks. Therefore, the proposed method is limited to the case of

$$b \geq 2\pi/T \tag{25}$$

D. Refinement

Using (24), we can estimate the phase parameters of SFM signals. However, it is worth noting that (8) is only an approximate equation; therefore, the obtained estimate is biased. However, the estimation can be used as a coarse estimation, and we can use the technique proposed in [1] for the refinement of the obtained coarse results.

III. Estimation of the PMC

Recall that we assume that PMC is a known constant in Section II. However, the PMC is unknown in practice and is required to be estimated. It is not easy to obtain the PMC satisfying (6), i.e., $D > ab^2$. Fortunately, we can take advantage of the framework of the QML estimator and use a one-dimensional (1-D) grid search to solve this problem.

Suppose that the signal $x(t)$ is uniformly sampled at the sampling interval T_s. To avoid the aliasing of PMF, the bandwidth of the PMF does not exceed the sampling rate, and thus is limited to

$$0 \leq DT \leq 2\pi/T_s \tag{26}$$

Using (26), we can obtain

$$0 \leq D \leq 2\pi/(TT_s) \tag{27}$$

Therefore, suppose that D is in the interval $[0, 2\pi/(TT_s)]$, we can use a 1-D grid search over the interval to select the PMC satisfying (6). The complete estimation algorithm for SFM signals is shown in Table I.

Remake: Note that (27) is only a restriction on D from the perspective of avoiding aliasing of PMF. However, D is unknown in fact, and we cannot give its interval. As a result, there are two cases. In the first case, if D is within the interval defined by (27), we can estimate it by a 1-D grid search mentioned above. In the second case, if D is not in the interval defined by (27), the proposed method will fail to obtain the parameter estimates of SFM signals.

IV. Computational Complexity

Now consider the computations required for the proposed algorithm. It consists of a phase modulation procedure, a calculating spectrum, a locating peaks step, and a refinement step.

TABLE I
ESTIMATION ALGORITHM FOR SFM SIGNALS

Inputs: The SFM signal $x[n]=x(nT_s), n=0,1,...,N-1$ and the set of the PMC $\mathbf{H}=\{2\pi k/(NTT_s) \mid k=0,1,...,N-1\}$ where $N=T/T_s$

1: ***For*** each PMC, $D \in \mathbf{H}$

2: Construct the PMF, $g[n]=\exp\left(jD(nT_s)^2/2\right)$.

3: Construct the modulated signal, $y[n]=x[n]g[n]$.

4: Calculate $z[n]=DFT(y[n])$ where $DFT(\bullet)$ denote the discrete Fourier transform.

5: Locate the AS peaks from $|z[n]|$.

6: Obtain the initial coarse estimate $\hat{\mathbf{a}}'_D$ using (24).

7: Apply the refinement approach in [1] to $\hat{\mathbf{a}}'_D$ to obtain the precise phase parameter estimate, $\hat{\mathbf{a}}^f_D=[\hat{a}^f_D \quad \hat{b}^f_D \quad \hat{c}^f_D]^T$.

8: Evaluate the QML function [1], [18].

$$J(D)=\left|\sum_{n=0}^{N-1}x[n]\exp\left(-j\hat{a}^f_D\sin\left(\hat{b}^f_D nT_s+\hat{c}^f_D\right)\right)\right|$$

9: ***End For***

10: Find optimal D maximizing $J(D)$

$$\hat{D}=\arg\max_D\left(J(D)\right)$$

Output: The final parameter estimation is obtained as $\hat{\mathbf{a}}=\hat{\mathbf{a}}^f_{\hat{D}}$

The phase modulation procedure requires $O(N)$ operations. The fast Fourier transform can obtain the modulated signal spectrum, and its complexity is $O(N\log N)$. The locating peaks step only requires N real additions, and its complexity is not higher than $O(N)$. The refinement step includes dechirping the signal with the coarse estimates, a simple moving average filter, and a basic unwrap. The three operations are linear in N, so its complexity is $O(N)$ [20], [22]. Therefore, the overall complexity of the proposed method is $O(N\log N)$. However, when we use a 1-D search to select the optimal PMC, the above procedures need to be repeated for each PMC. As a result, the computational complexity of the proposed method increases to $O(N^2\log N)$ from $O(N\log N)$. The estimator in [1], referred to as Igor Djurovic´ (ID) method, has complexity $O(N^3)$ [1], [18]. It is clear that the proposed method is computationally simpler than the ID algorithm.

V. NUMERICAL STUDY

In this section, simulation results are provided to evaluate the proposed estimator. We consider an SFM signal modeled as follows

$$x(t)=\exp\left(ja\sin\left(bt+c\right)\right),\ 0\leq t\leq T \qquad (28)$$

where $a=10\pi$, $b=0.06\pi$, $c=0$, and $T=50$s. The signal is corrupted by a zero-mean white Gaussian noise and uniformly sampled with the sampling interval T_s= 0.05 s having N = 1001 samples. For the sake of convenience, we refer to the proposed method as the phase modulation (PM) method. The proposed technique is compared with the ID estimator.

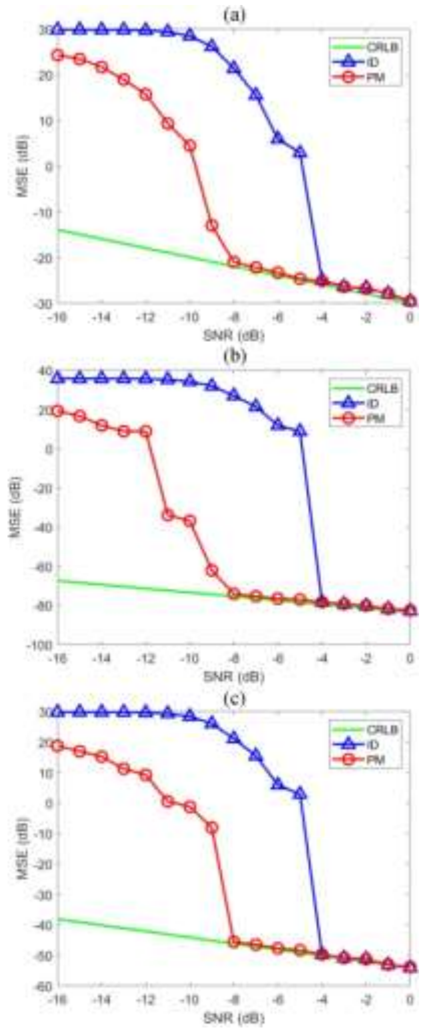

Fig. 2. MSEs of the phase parameters of the SFM obtained using the PM and ID. Thick solid line—CRLB; thick solid line with marker Δ—ID; thick solid line with marker ⓧ—PM.

The MSE of the estimated parameters has been used as a performance measure. The SNR range is SNR $\in[-16, 0]$ dB, added in increments of 1 dB. For each SNR, 500 Monte Carlo simulations are used to estimate the MSE of the estimates. Fig. 2 depicts the results obtained. As seen, the MSE value of the PM is close to the CRLB [23] when $SNR \geq -8$ dB, and the SNR threshold of the PM is better than that of the ID by 4 dB. Therefore, the proposed method can be used as an alternative approach for the estimation of SFM signals at low SNRs.

VI. CONCLUSION

We suggested a productive method for SFM signal estimation at low SNRs. The primary innovation of the suggested approach is that the original SFM signal's phase is modulated using LFM signals. By identifying the AS peaks of the modulated signal after phase modulation, we can quickly determine the phase parameters of the original SFM signal. The suggested technology has a lower SNR threshold than the current approaches, according to simulation data.

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